Abelian and non-Abelian quantum geometric tensor

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We propose a generalized quantum geometric tenor to understand topological quantum phase transitions, which can be defined on the parameter space with the adiabatic evolution of a quantum many-body system. The generalized quantum geometric tenor contains two different local measurements, the non-Abelian Riemannian metric and the non-Abelian Berry curvature, which are recognized as two natural geometric characterizations for the change in the ground-state properties when the parameter of the Hamiltonian varies. Our results show the symmetry-breaking and topological quantum phase transitions can be understood as the singular behavior of the local and topological properties of the quantum geometric tenor in the thermodynamic limit.

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I. INTRODUCTION

Quantum phase transitions (QPTs) are characterized by the *qualitative* changes in the ground-state properties when the parameter of a quantum many-body system varies. Different from classical phase transitions driven by the thermal fluctuations, QPTs occur at zero temperature and thus are driven entirely by the quantum fluctuations.[1] Traditionally, phase transitions are understood in the Landau-Ginzburg-Wilson paradigm with symmetry breaking and local order parameter. However, it has been revealed that this paradigm may not describe all possible orders and a new one *topological order*[2] needs to come into physics since the study of the integer and fractional quantum-Hall states. Whereas the preexistent symmetry in the Hamiltonian is absent, the phases can only be distinguished by certain topological invariants associated with the robust degeneracies of the ground state instead of any local order parameter. Historically, this was first pointed out in the integer quantum-Hall state and the corresponding topological invariant is recognized as the first Chern number.[3–5] Up to now, topological orders can be partially described by a new set of quantum numbers, such as ground-state degeneracy, quasiparticle fractional statistics, edge states, and topological entropy.[6] However, the classification of topological orders is still an open question.

More recently, the concept of quantum geometric tensor (QGT) (Ref. 7) based on differential geometry has been introduced to analyze continuous QPTs.[8,9] In this framework, the two approaches of the ground-state Berry phase and fidelity to witness QPTs are unified. It is shown that the underlying mechanism is the singular behavior of the QGT in the vicinity of the critical points. Specially, the real part of the QGT is a Riemannian metric defined over the parameter manifold while the imaginary part is the Berry curvature which flux give rise to the Berry phase. The Riemannian metric is recognized as the leading term of the fidelity[10] which is the overlap of two ground states associated to infinitesimally close parameters. Generally, the Riemannian metric will exhibit the divergent behavior in the quantum-critical region in the thermodynamic limit. In the approach of Berry phase, it was argued that a noncontractible ground-state Berry phase in the loop over the parameter space is associated to QPTs.[11] This fact indicates that the critical points associated to the divergence of the Berry curvature in the thermodynamic limit.[12] Particularly, a scaling analysis of this QGT in the vicinity of the critical points has been performed.[9] So far, the approach of QGT has been applied to detect the phases boundaries in various systems.[13]

What is underlying physical mechanism which endow the QGT with the Berry curvature and can it be extended to the non-Abelian case? Whether the QGT could be extended to a topological invariant and to distinguish the different topological phases not only the phases boundaries? In this paper, we present an adiabatic origin for the QGT and show a generalized non-Abelian QGT will naturally emerge from characterizing the distance between two neighbor states in the $U(n)$ vector bundle induced by the quantum adiabatic evolution of the degenerate system. In addition to a non-Abelian Riemannian metric as its symmetric part, the generalized QGT has also been equipped with a non-Abelian Berry curvature as its antisymmetric part. We demonstrate that this is due to the reduction in the parallel transport of states with the adiabatic evolution. Furthermore, we find its symmetric and antisymmetric parts correspond to two different local geometric measurements, which provide intrinsic characterizations for the change degree of the ground-state properties when the parameter of the Hamiltonian varies. As a general gauge covariant, the generalized QGT can be extended to the topological invariants Chern numbers because its antisymmetric part is just the instinct curvature tensor of the induced fiber bundles. This approach may provide a single framework to analyze a class of QPTs with topological orders.

II. ADIABATIC ORIGIN OF THE ABELIAN QUANTUM GEOMETRIC TENSOR

Let us consider a family of parameter-dependent Hamiltonian $\{H(\lambda)\}$ for a quantum many-body system. We require $H(\lambda)$ to depend smoothly on a set of parameters $\lambda = (\lambda^1, \lambda^2, \ldots) \in \mathcal{M}$ ($\mathcal{M}$ denotes the Hamiltonian parameters base manifold) and act over the Hilbert space $\mathcal{H}(\lambda)$. Especially, we are interested in the properties of the ground state $|\Psi_0(\lambda)\rangle$ of the eigenvalue $E_0(\lambda)$. To begin with, we restrict our discussion on the $U(1)$ case for convenience. Therefore the eigenvalue $E_0(\lambda)$ is required nondegenerate about all the parameters over $\mathcal{M}$, that is, the eigenspace $\mathcal{H}_{E_0}(\lambda)$ of $E_0(\lambda)$ is required to be one-dimensional. Now we associate each λ with a complex vector space $\mathcal{H}_{E_0}(\lambda)$ and thus forms a $U(1)$ line bundle. For the ground-state energy $E_0(\lambda)$, the subspace $\mathcal{H}_{E_0}(\lambda) := \mathrm{Span}\{|\psi_0(\lambda)\rangle\}$, where $|\psi_0(\lambda)\rangle$ is a complete orthonormal basis. In this basis, the ground state can be written as

$$|\Psi_0(\lambda)\rangle = C(\lambda)|\psi_0(\lambda)\rangle. \tag{1}$$

Obviously, the choice of basis is quite arbitrary and the component $C(\lambda)$ ($|C(\lambda)|^2=1$) of the ground state $|\Psi_0(\lambda)\rangle$ in different basis is related by a $U(1)$ local gauge transformation. Consider the distance between two neighbor states over the parameters base manifold $\mathcal{M}$, we have $dS := \|\Psi_0(\lambda+\delta\lambda) - \Psi_0(\lambda)\|$, which can be expanded to

$$dS^2 = \sum_{\mu,\nu} \langle \partial_\mu \Psi_0(\lambda) d\lambda^\mu | \partial_\nu \Psi_0 d\lambda^\nu(\lambda)\rangle, \tag{2}$$

where $\partial_\mu \Psi_0(\lambda) := \partial \Psi_0(\lambda)/\partial\lambda^\mu$. By using Eq. (1), the term $|\partial_\mu \Psi_0(\lambda)\rangle$ can be expanded as $|\partial_\mu \Psi_0(\lambda)\rangle = \partial_\mu C(\lambda)\cdot|\psi_0(\lambda)\rangle + C(\lambda)|\partial_\mu \psi_0(\lambda)\rangle$. Note that the term $C(\lambda)|\partial_\mu\psi_0(\lambda)\rangle$ will generally be mapped out of the subspace $\mathcal{H}_{E_0}(\lambda)$. Therefore we introduce the subspace $\mathcal{H}_{E_0}(\lambda)$ projection operator $\mathcal{P}(\lambda) := |\psi_0(\lambda)\rangle\langle\psi_0(\lambda)|$ and the complete projection operator 1. Then we have

$$|\partial_\mu \Psi_0(\lambda)\rangle = [\partial_\mu C(\lambda)|\psi_0(\lambda)\rangle + C(\lambda)\mathcal{P}(\lambda)|\partial_\mu\psi_0(\lambda)\rangle] + C(\lambda)[1 - \mathcal{P}(\lambda)]|\partial_\mu\psi_0(\lambda)\rangle. \tag{3}$$

Now we introduce the quantum adiabatic evolution to the system and suppose that the dim $\mathcal{H}_{E_0}(\lambda)$ does not depend on the parameter λ, and particularly, demand that during the evolution of the system there are no level crossings. Under the quantum adiabatic limit, the evolution of the ground state $|\Psi_0(\lambda)\rangle \in \mathcal{H}_{E_0}(\lambda)$ to $|\Psi_0(\lambda+\delta\lambda)\rangle \in \mathcal{H}_{E_0}(\lambda+\delta\lambda)$ will undergo a parallel transport in the sense of Levi-Civitá from λ to $\lambda+\delta\lambda$ on the base manifold $\mathcal{M}$. We have

$$|D_\mu \Psi_0(\lambda)\rangle = 0, \tag{4}$$

where $|D_\mu\Psi_0(\lambda)\rangle$ denote the covariant derivative of $|\Psi_0(\lambda)\rangle$ in the $U(1)$ line bundle, which can be written in the basis of $\{|\psi_0(\lambda)\rangle\}$ as $|D_\mu\Psi_0(\lambda)\rangle = \partial_\mu C(\lambda)\cdot|\psi_0(\lambda)\rangle + C(\lambda)\cdot\mathcal{P}(\lambda)\cdot|\partial_\mu\psi_0(\lambda)\rangle$. It is clear that the term of $\mathcal{P}(\lambda)\cdot|\partial_\mu\psi_0(\lambda)\rangle$ play the role of connection. We have $\mathcal{P}(\lambda)\cdot|\partial_\mu\psi_0(\lambda)\rangle = -i\mathcal{A}_\mu|\psi_0(\lambda)\rangle$, where the connection coefficients $\mathcal{A}_\mu =: i\langle\psi_0|\partial_\mu\psi_0\rangle$ is just the Berry connection. Substituting Eqs. (3) and (4) to Eq. (2), we have $\langle\partial_\mu\Psi_0(\lambda)|\partial_\nu\Psi_0(\lambda)\rangle = \langle\partial_\mu\psi_0(\lambda)|[1-\mathcal{P}(\lambda)]|\partial_\nu\psi_0(\lambda)\rangle$. Thus we get a Hermitian metric tensor

$$Q_{\mu\nu} := \langle\partial_\mu\psi_0(\lambda)|[1-\mathcal{P}(\lambda)]|\partial_\nu\psi_0(\lambda)\rangle. \tag{5}$$

This is the adiabatic origin of the Abelian quantum geometric tensor. The Hermitian metric $Q_{\mu\nu}$ defines a positive definite symmetric Riemannian metric $g_{\mu\nu} := (Q_{\mu\nu}+Q^*_{\mu\nu})/2$ [in the $U(1)$ case, $g_{\mu\nu} = \Re Q_{\mu\nu}$] and then the distance can be written as

$$dS^2 = \sum_{\mu,\nu} g_{\mu\nu} d\lambda^\mu d\lambda^\nu. \tag{6}$$

We can also associate to $Q_{\mu\nu}$ a two form $F = \Sigma_{\mu,\nu} F_{\mu\nu} d\lambda^\mu \wedge d\lambda^\nu$, where $F_{\mu\nu} := i(Q_{\mu\nu} - Q^*_{\mu\nu})$ [in the $U(1)$ case, $F_{\mu\nu} = -2\Im Q_{\mu\nu}$], which is nothing but the Berry curvature. Furthermore we can get the relation

$$Q_{\mu\nu} = g_{\mu\nu} - \frac{i}{2}F_{\mu\nu}. \tag{7}$$

III. NON-ABELIAN QUANTUM GEOMETRIC TENSOR

Here we point out that the QGT can be extended to a more general case in which a non-Abelian QGT can be naturally defined on the $U(N)$ vector bundle induced by the adiabatic development of the quantum degenerate system. This approach is very similar to the $U(1)$ case but now the subspace $\mathcal{H}_{E_0}(\lambda)$ is a *ND* Hilbert space. For the ground energy $E_0(\lambda)$, the eigenspace is $\mathcal{H}_{E_0}(\lambda) := \mathrm{Span}\{|\psi_i(\lambda)\rangle\}_{i=1}^N$. A ground state $|\Psi_0(\lambda)\rangle \in \mathcal{H}_{E_0}(\lambda)$ can be expanded as $|\Psi_0(\lambda)\rangle = \Sigma_{i=1}^N C_i(\lambda)|\psi_i(\lambda)\rangle$ and the wave function $[C_1(\lambda), C_2(\lambda), \ldots, C_N(\lambda)]^T$ represented in different basis is related by a $U(N)$ local gauge transformation. The distance between two neighbor states over $\mathcal{M}$ is $dS^2 = \Sigma_{\mu,\nu}\langle\partial_\mu\Psi_0(\lambda)d\lambda^\mu|\partial_\nu\Psi_0 d\lambda^\nu(\lambda)\rangle$, in which the term $|\partial_\mu\Psi_0(\lambda)\rangle$ can be decomposed as $|\partial_\mu\Psi_0(\lambda)\rangle = |D_\mu\Psi_0(\lambda)\rangle + \Sigma_{i=1}^N C_i(\lambda)[1-\mathcal{P}(\lambda)]|\partial_\mu\psi_i(\lambda)\rangle$, where $\mathcal{P}(\lambda) := \oplus_{j=1}^N |\psi_j(\lambda)\rangle\langle\psi_j(\lambda)|$ is the projection operator and $|D_\mu\Psi_0(\lambda)\rangle = \Sigma_{i=1}^N \partial_\mu C_i(\lambda)\cdot|\psi_i(\lambda)\rangle + C_i(\lambda)\cdot\mathcal{P}(\lambda)\cdot|\partial_\mu\psi_i(\lambda)\rangle$ denote the covariant derivative of $|\Psi_0(\lambda)\rangle$ in the $U(n)$ vector bundle.

Now let us introduce the quantum adiabatic evolution to the degenerate system, therefore $|D_\mu\Psi_0(\lambda)\rangle = 0$, which means the ground state $|\Psi_0(\lambda)\rangle$ changing to $|\Psi_0(\lambda+\Delta\lambda)\rangle$ along an arbitrary path $\Gamma(\lambda)$ on the parameter base manifold $\mathcal{M}$ will undergo a parallel transport. Note that the term $\mathcal{P}(\lambda)\cdot|\partial_\mu\psi_m(\lambda)\rangle = -i\Sigma_{l=1}^N \mathcal{A}_\mu^{lm}|\psi_l(\lambda)\rangle$, here $\mathcal{A}_\mu^{lm} = i\langle\psi_l|\partial_\mu\psi_m\rangle$ is known as the Wilczek-Zee connection,[14] which is a matrix element of the non-Abelian gauge potential. By using the above relations, we can derive the distance between two neighbor states over $\mathcal{M}$ as

$$dS^2 = \sum_{\mu,\nu}\langle\partial_\mu\Psi_0(\lambda)d\lambda^\mu|\partial_\nu\Psi_0 d\lambda^\nu(\lambda)\rangle = \sum_{\mu\nu}\left[(C_1^* \ \cdots \ C_n^*)Q_{\mu\nu}\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}\right] d\lambda^\mu d\lambda^\nu, \tag{8}$$

where $Q_{\mu\nu}$ is a $N\times N$ Hermitian matrix. We define this quantity as the non-Abelian QGT. Its matrix element is given by

$$Q^{ij}_{\mu\nu} := \langle\partial_\mu\psi_i(\lambda)|[1-\mathcal{P}(\lambda)]|\partial_\nu\psi_j(\lambda)\rangle. \tag{9}$$

Now we can derive the corresponding relation to Eq. (7) in the non-Abelian case, i.e., $Q_{\mu\nu} = g_{\mu\nu} - \frac{i}{2}F_{\mu\nu}$, where $g_{\mu\nu} = (Q_{\mu\nu} + Q^\dagger_{\mu\nu})/2$ and $F_{\mu\nu} = i(Q_{\mu\nu} - Q^\dagger_{\mu\nu})$. Now all the components have been extended to the matrix form, the symmetry and antisymmetry part correspond to the non-Abelian Riemannian metric and non-Abelian Berry curvature, respectively.

IV. GEOMETRIC CHARACTERIZATIONS OF GROUND-STATE PROPERTIES

We say that a QPT occurs if the *qualitative* properties (no matter local or topological) of the ground-state change when the parameter of the Hamiltonian varies. However, it is hard to give an explicit and general definition for in what degree a change of the properties of the ground state should be regarded as a *qualitative* change. The concept of the QGT will shed light on this problem. Note that dS represents the distance between two neighbor states located on λ and $\lambda+d\lambda$ of the Hamiltonian parameters base manifold. Obviously, the quantity $(\Delta S/\Delta\lambda)^2$ will be a good measurement about the degree of the change in ground state respect to the change of the parameter. By using Eq. (6) or (8), we have

$$\lim_{\Delta\lambda\to 0}\frac{\Delta S^2}{\Delta\lambda^2}=\sum_{\mu,\nu} g_{\mu\nu}\frac{\partial\lambda^\mu}{\partial\lambda}\otimes\frac{\partial\lambda^\nu}{\partial\lambda}. \tag{10}$$

By the way, we would like to point out that the scalar $\chi_{FS} := \lim_{\Delta\lambda\to 0}\Delta S^2/\Delta\lambda^2$ in the $U(1)$ case provides a purely geometric definition for the *fidelity susceptibility*.[15] Note that dS is the distance in the orthocomplement space of the subspace $\mathcal{H}_{E_0}(\lambda)$ so χ_{FS} is clearly not an intrinsic geometric quantity for the induced fiber bundles.

However, there still exist an intrinsic geometric measurement. Consider the parallel transport of ground state around a small loop on the parameter base manifold $\mathcal{M}$, there generally exist a $U(n)$ rotation of the ground state in the nD internal subspace $\psi_{Final}=\mathcal{P}\exp[i\oint_C\mathcal{A}_\mu d\lambda^\mu]\psi_{Initial}$, where $\mathcal{P}$ is the path-ordering operator and $\mathcal{A}_\mu$ is the Wilczek-Zee connection. In the first order approximation, we have $\mathcal{P}\exp[i\oint_C\mathcal{A}_\mu d\lambda^\mu]\approx 1+i\Sigma_{\mu,\nu}F_{\mu\nu}\sigma^{\mu\nu}$, where $F_{\mu\nu}$ is the non-Abelian Berry curvature and $\sigma^{\mu\nu}$ is the area element $d\lambda^\mu\wedge d\lambda^\nu$ of the infinitesimal surface σ, whose boundary is

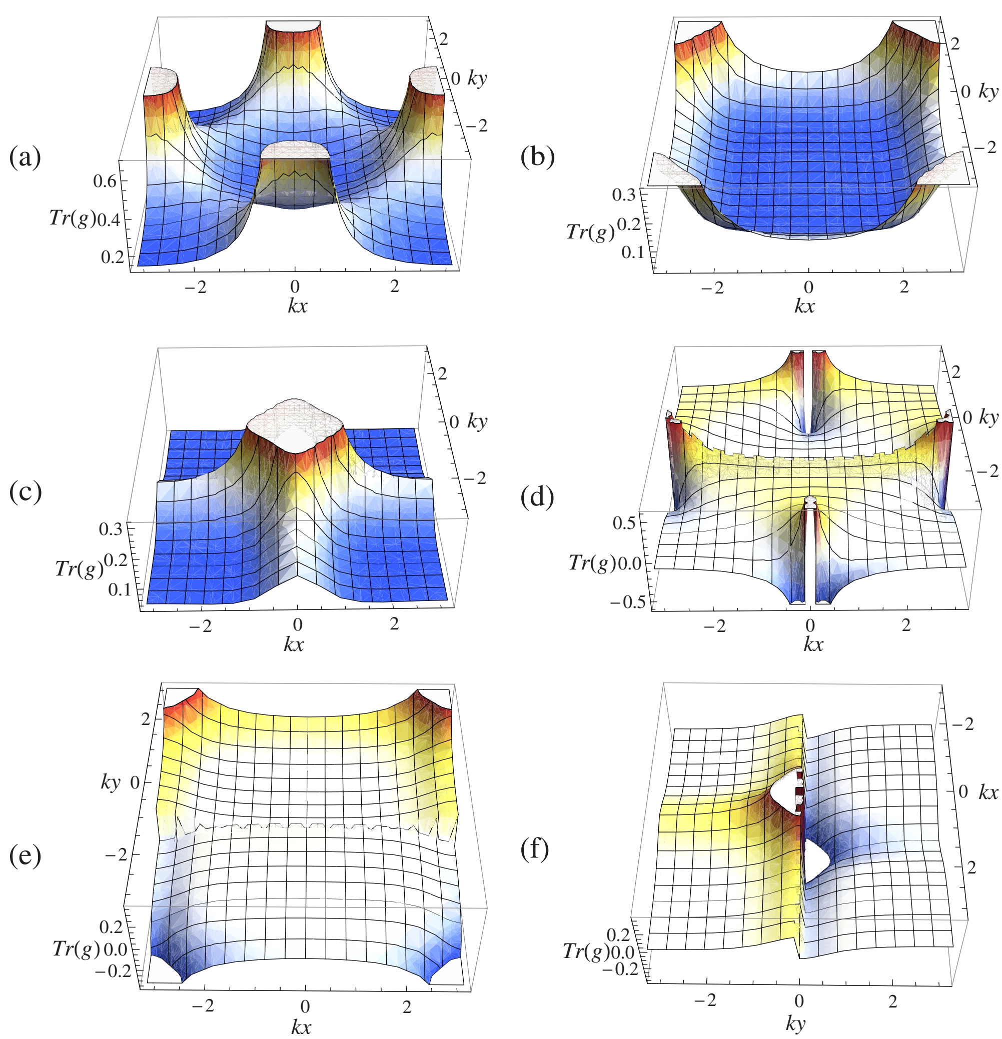

FIG. 1. (Color online) (a) The trace of the Riemannian metric g as a function of the k_x, k_y, in which the parameter $m=0$, (b) $m=2$, (c) $m=-2$, (d) the Berry curvature F as a function of the k_x, k_y, in which the parameter $m=0$, (e) $m=2$, and (f) $m=-2$.

the curve C. Now we can define a local geometric measurement depend on $\lambda \in \mathcal{M}$ as follows:

$$\lim_{\sigma\to 0}\frac{\mathcal{P}\exp\left[i\int_C \mathcal{A}_\mu d\lambda^\mu\right]-1}{\sigma}=\sum_{\mu,\nu}F_{\mu\nu}\frac{\partial\lambda^\mu}{\partial\lambda}\wedge\frac{\partial\lambda^\nu}{\partial\lambda}. \quad (11)$$

Note that the above definition is equivalent to $(\psi_{Final}-\psi_{Initial})\psi_{Initial}^{-1}\sigma^{-1}$ with $\sigma\to 0$, which is the measurement of the intensity of relative change in the state along a infinitesimal loop. In compare to Eq. (10), this is purely an instinct definition. We find that the non-Abelian QGT make a unification of the above two approaches through the relation $Q_{\mu\nu}=g_{\mu\nu}-iF_{\mu\nu}/2$. Note that the instinct curvature of the induced fiber bundles $F_{\mu\nu}=i(Q_{\mu\nu}-Q_{\mu\nu}^\dagger)$ can be naturally associated to the topological invariant Chern numbers, i.e., the first Chern number can be calculated as

$$C_1=\frac{i}{2\pi}\int_{\mathcal{M}}(Q_{\mu\nu}-Q_{\mu\nu}^\dagger)d\lambda^\mu\wedge d\lambda^\nu. \quad (12)$$

Historically, a two-band model with a nonzero Chern number was first proposed as by Haldane,[16] which was a honeycomb lattice model with imaginary next-nearest-neighbor hopping. Haldane find the two different symmetries breaking about the space reflection and the time reversal can be classified by the Chern numbers which reflect the topology of the ground state.[17] Recently, Hatsugai[18] proposed a non-Abelian Chern number for a ground state multiplet with a possible degeneracy and found the system to be a topological insulator when the energies of the multiplet are well separated from the above.

Here we choose a two-band model as an example because it is one of the simplest models that exhibit the topological nontrivial states. This model was first introduced by Qi[19] and can be physically realized in $Hg_{1-x}Mn_xTe/Cd_{1-x}Mn_xTe$ quantum wells with a proper amount of Mn spin polarization.[20] The system under consideration is a two-dimensional (2D) lattice model on a two torus T^2. The Hamiltonian is generally given by $H(k)=\varepsilon(k)1+\sum_{\alpha=1}^{3}d_\alpha(k)\sigma^\alpha$, where 1 is the 2×2 identity matrix and σ^α the three Pauli matrix. The diagonalization of $H(k)$ is straightforward and the eigenvalues can be written as $E_\pm(k)=\varepsilon(k)\pm\sqrt{\sum_{\alpha=1}^{3}d_\alpha^2(k)}$ and eigenvectors as $\Psi_+=(\cos\theta/2, e^{i\Phi}\sin\theta/2)^T$ and $\Psi_-=(-\sin\theta/2, e^{i\Phi}\cos\theta/2)^T$, where $\theta=\arccos d_3(k)/\sqrt{d_1^2(k)+d_2^2(k)+d_3^2(k)}$ and $\Phi=\arctan d_1(k)/\sqrt{d_1^2(k)+d_2^2(k)}$. In this model, the coefficients are given by $\varepsilon(k)=0$, $d_1=\sin k_x$, $d_2=\sin k_y$, and $d_3=m+\cos k_x+\cos k_y$. In the thermodynamic limit, the fiber can be chosen as the single-particle wave function and the base manifold is the 2D momentum space $\boldsymbol{k}=(k_x,k_y)$. Then QGT can be obtained by substituting Ψ_- into Eq. (5), we have $Q_{xy}=(\partial_{kx}\theta\partial_{ky}\theta+\partial_{kx}\Phi\partial_{ky}\Phi\sin^2\theta)/4+i\sin\theta(\partial_{kx}\Phi\partial_{ky}\theta-\partial_{ky}\Phi\partial_{kx}\theta)/4$. The corresponding Riemannian metric and Berry curvature can be obtained by using the relation $g_{xy}=\Re Q_{xy}$ and $F_{xy}=-2\Im Q_{xy}$, we have $g_{xy}=(\partial_{kx}\theta\partial_{ky}\theta+\partial_{kx}\Phi\partial_{ky}\Phi\sin^2\theta)/4$ and $F_{xy}=-\sin\theta(\partial_{kx}\Phi\partial_{ky}\theta-\partial_{ky}\Phi\partial_{kx}\theta)/2$. We now analyze the behavior of the Riemannian metric $g_{\mu\nu}$ and Berry curvature $F_{\mu\nu}$ when the parameter m is varied. The numerical results are presented in Fig. 1, in which the divergence behavior are shown either in g_{xy} or F_{xy} at some region in $\boldsymbol{k}$ space when the parameter m takes the values of -2, 0, 2. This indicate a vanishing energy gap. The critical points can be clear shown through integrating out k_x, k_y of the Riemannian metric $g_{xy}(m)$. As shown in Fig. 2(a), we can see clearly a crown shape curve with three divergent points at $m=-2$, $m=0$, and $m=2$, respectively. In order to distinguish the different topological phases, we can calculate the topological invariant by using Eq. (12)

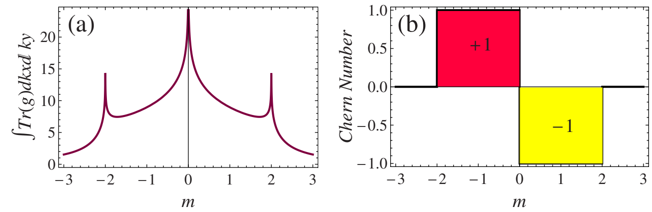

FIG. 2. (Color online) (a) Integrate out k_x, k_y of the Riemannian metric g as a function of m and (b) the first Chern number $c1$ as a function of m.

$$C_1=\begin{cases}1 & \text{if} \quad -2<m<0\\ -1 & \text{if} \quad 0<m<2\\ 0 & \text{otherwise},\end{cases} \quad (13)$$

which is the first Chern number of the induced $U(1)$ line bundle. The phase diagram is plotted in Fig. 2(b).

V. CONCLUSIONS

We have presented an adiabatic origin for the generalized QGT and shown a non-Abelian QGT can naturally emerge from measuring the distance between two neighbor states in the $U(N)$ vector bundle induced by the adiabatic evolution of the quantum degenerate system. In addition, we introduced two different local geometric measurements based on an intuitive physical picture, i.e., $\lim_{\Delta\lambda\to 0}\Delta S^2/\Delta\lambda^2$ and $\lim_{\sigma\to 0}[\mathcal{P}\exp(i\oint_C\mathcal{A}_\mu d\lambda^\mu)-1]/\sigma$ to characterize the change degree of the ground-state when the parameter varies. As the main results of this work, we demonstrated the two different measurements can be unified in the generalized non-Abelian QGT as its symmetric and antisymmetric parts, respectively. We showed the symmetry-breaking QPTs and a class of topological QPTs can be understood as the singular behavior of the local and topological properties of the QGT in the thermodynamic limit. This approach shed light on constructing a single framework to analyze QPTs with topological orders.

Note added. After this manuscript has been submitted, we noticed that a related work appeared,[21] which similarly studies the geometry of quantum adiabatic evolution and generalized the treatments of the quantum geometric tensor to allow for degeneracy.

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